Semantics and Proof Theory of the Epsilon Calculus

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Outline

1. Introduction
2. The Epsilon Calculus
3. Subclassical Logics (joint work with M. Baaz)
4. Proof Theory for Epsilon Calculus
5. Conclusion

What is the Epsilon Calculus?

- Formalization of logic without quantifiers but with the ε-operator.
- If $A(x)$ is a formula, then $\varepsilon_x A(x)$ is an ε-term.
- Intuitively, $\varepsilon_x A(x)$ is an indefinite description: $\varepsilon_x A(x)$ is some x for which $A(x)$ is true.
- ε can replace $\exists$: $\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x))$
- Axioms of ε-calculus:
  - Propositional tautologies
  - (Equality schemata)
  - $A(t) \to A(\varepsilon_x A(x))$
- Predicate logic can be embedded in ε-calculus.

Why Should You Care?

- Epsilon calculus is of significant historical interest.
  - Origins of proof theory
  - Hilbert's Program
- Alternative basis for fruitful proof-theoretic research.
  - Epsilon Theorems and Herbrand's Theorem: proof theory without sequents
  - Epsilon Substitution Method: yields functionals, e.g.,

$$\vdash \forall x \exists y\, A(x, y) \rightsquigarrow \forall n \colon \vdash A(n, f(n))$$

- Interesting Logical Formalism
  - Trade logical structure for term structure.
  - Suitable for proof formalization.
- Other Applications:
  - Applications in linguistics (choice functions, anaphora).
  - Connections to Fine's "arbitrary object" theory.
  - Propositions-as-types for dynamic linking.

Epsilon Substitution and Epsilon Theorems

Two approaches to consistency proofs in the ε-calculus:

1. **Epsilon Substitution:** For every epsilon term $\varepsilon_x A(x)$, find a numerical substitution; i.e., interpret εs as particular numbers.
    - Specific to arithmetical theories.
    - Developed by Ackermann (1924, 1940), von Neumann (1927)
2. **Epsilon Theorems:** Eliminate epsilon terms "directly" from a proof using proof transformations.
    - Can be applied to any quantifier-free theory.
    - Difficult to extend to arithmetic (induction).
    - Epsilon theorems have other applications as well (e.g., Herbrand's theorem)

The Epsilon Calculus: Syntax

- $\wedge$, $\vee$, $\rightarrow$, …
- If $A(x)$ is a formula then $\forall x\, A(x)$ and $\exists x\, A(x)$ are formulas.
- If $A(x)$ is a formula, then $\varepsilon_x A(x)$ is a term.
- An ε-term $p \equiv \varepsilon_x A(x; y_1, \dots, y_n)$ is the ε-type of an ε-term e if
  - the y_i are all immediate subterms,
  - every y_i has exactly one occurrence, and
  - $e \equiv \varepsilon_x A(x; t_1, \dots, t_n)$.
- Every ε-term a substitution instance of an ε-matrix.

Extensional Semantics

- Interpretation: $M = \langle D, \Phi, s \rangle$
  - $D \neq \emptyset$ is the domain
  - ^{M}: interpretation of function and predicate symbols
  - $s : \mathit{Var} \to D$: variable assignment
  - Φ an extensional choice function
- Extensional choice function:

$$\Phi(S) \in S \quad \text{if} \quad S \neq \emptyset$$

- Valuation of ε-terms $\varepsilon_x A(x)$

$$\mathrm{val}_{M,\Phi,s}(\varepsilon_x A(x)) = \Phi(\hat{x}[A(x)]_{M,\Phi,s})$$

where $\hat{x}[A(x)]_{M,\Phi,s} = \{d \in D : M, \Phi, s[d/x] \models A(x)\}$.

Intensional Semantics

- Interpretation: $M = \langle D, \Phi, s \rangle$
  - $D \neq \emptyset$ is the domain
  - ^{M}: interpretation of function and predicate symbols
  - $s : Var \to D$: variable assignment
  - Ψ an intensional choice function
- Intensional choice function:

$$\Psi(S, p, d_1, \ldots, d_n) \in S \quad \text{if} \quad S \neq \emptyset$$

- Valuation of ε-terms $\varepsilon_x A(x) = p(t_1, \ldots, t_n)$ with type $p = \varepsilon_x A'(x; y_1, \ldots, y_n)$:

$$\varepsilon_x A(x)^M = \Phi(\hat{x}[A(x)]_{M,\Psi,s}, p, t_1^M, \ldots, t_n^M)$$

Axiomatisation of the Epsilon Calculus

- EC (axioms of the elementary calculus): all propositional tautologies
- EC_ε (the pure epsilon calculus): add to EC all substitution instances of

$$A(t) \to A(\varepsilon_x A(x)) . \qquad (1)$$

  An axiom of the form (1) is called a critical formula.
- PC (the predicate calculus), PC_ε (extended predicate calculus): EC and EC_ε, respectively, together with all instances of $A(t) \to \exists x\, A(x)$ and $\forall x\, A(x) \to A(t)$ in the respective language.

Completeness

- Elementary calculus/extended predicate calculus complete for intensional semantics
- EC_ε with identity axioms plus ε-identity schema

$$t = u \to \varepsilon_x A(x; s_1 \dots t \dots s_n) = \varepsilon_x A(x; s_1 \dots u \dots s_n)$$

  complete for intensional semantics including =
- EC_ε with identity, ε-identity, and ε-extensionality schema

$$\forall x (A(x) \leftrightarrow B(x)) \to \varepsilon_x A(x) = \varepsilon_x B(x)$$

  complete for extensional semantics.

Embedding PC in EC_ε

Map $^\varepsilon$ of expressions in $L(PC_\varepsilon)$ to expressions in $L(EC_\varepsilon)$ as follows:

- $x^\varepsilon = x$
- $P(t_1, \ldots, t_n)^\varepsilon = P(t_1^\varepsilon, \ldots, t_n^\varepsilon)$
- $(\neg A)^\varepsilon = \neg A^\varepsilon$
- $(A \lor B)^\varepsilon = A^\varepsilon \lor B^\varepsilon$
- $(A \land B)^\varepsilon = A^\varepsilon \land B^\varepsilon$
- $(A \to B)^\varepsilon = A^\varepsilon \to B^\varepsilon$
- $(\varepsilon_x A(x))^\varepsilon = \varepsilon_x\, A(x)^\varepsilon$
- $(\exists x\, A(x))^\varepsilon = A^\varepsilon(\varepsilon_x A(x)^\varepsilon)$
- $(\forall x\, A(x))^\varepsilon = A^\varepsilon(\varepsilon_x \neg A(x)^\varepsilon)$

The Embedding Lemma

- A^ε is of the form:

$$[A(t) \to \exists x\, A(x)]^\varepsilon \equiv A^\varepsilon(t^\varepsilon) \to A^\varepsilon(\varepsilon_x A(x)^\varepsilon)\ ,$$

  which is a critical formula.

- A^ε is of the form:

$$[\forall x\, A(x) \to A(t)]^\varepsilon \equiv A^\varepsilon(\varepsilon_x \neg A(x)) \to A^\varepsilon(t^\varepsilon)$$

  This is the contrapositive of, and hence provable from, the critical formula

$$\neg A^\varepsilon(t^\varepsilon) \to \neg A^\varepsilon(\varepsilon_x \neg A(x))$$

The First Epsilon Theorem

First Epsilon Theorem

If A is a formula without bound variables (no quantifiers, no epsilons) and $\mathrm{PC}^{\varepsilon} \vdash A$ then $\mathrm{EC} \vdash A$.

Extended First Epsilon Theorem

If $\exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n)$ is a purely existential formula containing only the bound variables x_1, …, x_n, and

$$\mathrm{PC}^{\varepsilon} \vdash \exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n),$$

then there are terms t_{ij} such that

$$\mathrm{EC} \vdash \bigvee_i A(t_{i1}, \ldots, t_{in}).$$

Herbrand Theorem

Herbrand Theorem for $\exists_1$

If $\exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n)$ is a purely existential formula

$$\mathrm{PC} \vdash \exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n),$$

then there are terms t_{ij} such that

$$\mathrm{EC} \vdash \bigvee_i A(t_{i1}, \ldots, t_{in}).$$

From the last formula, the original formula can be proved in PC.

- Can be extended to prenex formulas (by "Herbrandization")
- Can be extended to all formulas, since PC proves every formula equivalent to prenex form.

Extended First Epsilon Theorem

Extended First Epsilon Theorem

Suppose $E(e_1, \ldots, e_m)$ is a quantifier-free formula containing only the ε-terms e_1, …, e_m, and

$$\mathrm{EC}_\varepsilon \vdash_\pi E(e_1, \ldots, e_m)\ ,$$

then there are ε-free terms t^i_j such that

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \ldots, t^i_m)$$

where $n \leq 2^{2^{\cdot^{\cdot^{\cdot^{2^{3\cdot\mathrm{cc}(\pi)}}}}}} \Big\}$ stack of $3 \cdot \mathrm{cc}(\pi)$ 2's.

Superintuitionistic Logics

- In classical logic, $\exists$ and $\forall$ are interdefinable
- Not true in subclassical logics such as intuitionistic logic
- Epsilon operator seems intuitively related to choice, so intuitionistically suspect
- So: what happens when ε added to a superintuitionistic logic?

Interdefinability of ∀ and ∃

- In classical logic:

$$\neg\exists x\, \neg A(x) \leftrightarrow \forall x\, A(x)$$
$$\neg\neg A(\varepsilon_x \neg A(x)) \leftrightarrow A(\varepsilon_x \neg A(x))$$

- $\rightarrow$ fails in intuitionistic logic

Solution: ε and τ

- Introduce dual operator τ: $\tau_x A(x)$
- Critical formulas now:

$$A(t) \to A(\varepsilon_x A(x)) \text{ and}$$
$$A(\tau_x A(x)) \to A(t)$$

- $\varepsilon\tau$-translation just like ε-translation, except for:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x))$$
$$\forall x\, A(x) \Leftrightarrow A(\tau_x A(x))$$

Effect of $\varepsilon\tau$ on Propositional Level

- In classical logic, addition of ε is conservative.
- Question: Does addition of ε and τ to superintuitionistic logic have effect on theorems?
- Results by Bell and DeVidi suggest yes: under certain assumptions, even excluded middle $A \vee \neg A$ becomes provable.
- However, these results rely on presence of $=$ and need axioms.
- What about pure logic?
    - No effect on propositional level.
    - All quantifier shifts become provable.

$\varepsilon\tau$ Conservative for Propositional Logic

Conservativity of $\varepsilon\tau$

If $A_1, \dots, A_n \vdash_{L^{\varepsilon\tau}} B$, then $A_1^s, \dots, A_n^s \vdash B^s$, provided

- removing quantfiers results in propositional theorems
- $A \to A$ is provable

Quantifier Shifts

$$(\forall\vee)\quad \forall x(A \vee B) \to (\forall x\, A \vee B)$$
$$(A(\tau_x(A \vee B)) \vee B) \to (A(\tau_x A) \vee B)$$
$$(\to)\exists\quad (B \to \exists x\, A) \to \exists x(B \to A)$$
$$(B \to A(\varepsilon_x A)) \to (B \to A(\varepsilon_x(B \to A)))$$
$$\exists(\to)\quad (\forall x\, A \to B) \to \exists x(A \to B)$$
$$(A(\tau_x A) \to B) \to (A(\varepsilon_x(A \to B)) \to B)$$

In each case, x is not free in B.

Epsilon Theorem in Subclassical Logics

- In intuitionistic and Gödel logics, there are no (usual) prenex normal forms
- However, in intuitionistic and Gödel $\varepsilon\tau$-calculi, all quantifier shifts are provable, so every formula is equivalent to a prenex formula
- Provability of
  - "Herbrand form" from prenex formula, and
  - of prenex formula from Herbrand disjunction

  require only weak assumptions on the logic (true in intuitionistic and Gödel logic)
- Question: extended epsilon theorem true in intuitionistic and Gödel $\varepsilon\tau$-calculi?

No Herbrand Theorem in Subclassical $\varepsilon\tau$-Logics

Theorem

Suppose $L^{\varepsilon\tau}$ has the extended first epsilon theorem, $\vdash_L A \to A$, and in L, $\vee$ is provably commutative, associative, and idempotent, and has weakening $(A \to (A \vee B))$. Then

$$L \vdash (A_1 \to A_2) \vee \ldots \vee (A_k \to A_{k+1})$$

for some k.

Corollary

Intuitionistic and Gödel $\varepsilon\tau$-calculi do not have the extended first epsilon theorem.

Summary of Results

- Adding ε (and τ) to intuitionistic and intermediate logics has
  - no effect on propositional level
  - results in all quantifier shifts becoming provable
- Epsilon elimination is much more problematic than in classical logic
  - Logics where forking sentences are all invalid (i.e., all logics with frames of unbounded size) cannot have extended epsilon theorem
  - This includes in particular intuitionistic and (infinite-valued) Gödel $\varepsilon\tau$-logics
  - In $\varepsilon\tau$-logics of k-valued Gödel logics, epsilon theorem holds

A One-sided Sequent Calculus

- Axiom: $A, \neg A$
- Rules:

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \wedge R \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \wedge L \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A} \neg\neg$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} \vee R \qquad \frac{\Gamma, \neg A \quad \Gamma, \neg B}{\Gamma, \neg(A \vee B)} \vee L \qquad \frac{\Pi, A \quad \Lambda, \neg A}{\Pi, \Lambda} \mathit{cut}$$

$$\frac{\Gamma, A(t)}{\Gamma, \exists x\, A(x)} \exists R \qquad \frac{\Gamma, \neg A(x)}{\Gamma, \neg\exists x\, A(x)} \exists L$$

$$\frac{\Gamma, A(x)}{\Gamma, \forall x\, A(x)} \forall R \qquad \frac{\Gamma, \neg A(t)}{\Gamma, \neg\forall x\, A(x)} \forall L$$

Leisenring's Sequent Calculus

$$\frac{\Gamma, A(t)}{\Gamma, \exists x\, A(x)}\ \exists R \qquad \frac{\Gamma, \neg A(\varepsilon_x A(x))}{\Gamma, \neg\exists x\, A(x)}\ \exists L$$

$$\frac{\Gamma, A(\varepsilon_x \neg A(x))}{\Gamma, \forall x\, A(x)}\ \forall R \qquad \frac{\Gamma, \neg A(t)}{\Gamma, \neg\forall x\, A(x)}\ \forall L$$

No eigenvariable conditions!

Completeness: Deriving Critical Formulas

- Derives everything EC_ε derives:

$$\dfrac{\dfrac{\neg A(t), A(t)}{\neg A(t), \exists x\, A(x)}\ \exists R \qquad \dfrac{\neg A(\varepsilon_x A(x)), A(\varepsilon_x A(x))}{\neg \exists x\, A(x), A(\varepsilon_x A(x))}\ \exists L}{\neg A(t), A(\varepsilon_x A(x))}\ \mathit{cut}$$

- Obviously has no cut-free proof
- Hence, Leisenring's system not cut-free complete

Maehara's Sequent Calculus

- Axioms: $\neg A, A \qquad \neg A(t), A(\varepsilon_x A(x))$
- Complete, since additional axioms allow derivation of critical formulas.
- However, not cut-free complete.

Maehara's System Not Cut-free Complete

- Converse of critical formulas derivable:

$$\dfrac{\neg\neg A(t), \neg A(\varepsilon_x \neg A(x)) \quad \neg A(t), A(t)}{\neg A(\varepsilon_x \neg A(x)), A(t)}\ cut$$

- But obviously no cut-free proof

The Mints-Yasuhara System

- Additional rule:

$$\frac{\Gamma, \Delta(\varepsilon_x A(x)), \neg A(\varepsilon_x A(x)) \quad \Gamma, A(t)}{\Gamma, \Delta(\varepsilon_x A(x))} \; \varepsilon_1$$

$\Delta(\varepsilon_x A(x))$ must be not empty.

- Derives critical formulas:

$$\frac{\dfrac{A(\varepsilon_x A(x)), \neg A(\varepsilon_x A(x))}{\underbrace{\neg A(t)}_{\Gamma}, \underbrace{A(\varepsilon_x A(x))}_{\Delta}, \neg A(\varepsilon_x A(x))}\; w \qquad \underbrace{\neg A(t)}_{\Gamma}, A(t)}{\neg A(t), A(\varepsilon_x A(x))} \; \varepsilon_1$$

Gentzen-style Cut Elimination

- Main induction on cut length, i.e., height of tree above uppermost cut.
- Induction step: permute cut upward.
- For instance, replace proof ending in cut

$$\dfrac{\dfrac{\vdots\, D}{\Pi, A} \qquad \dfrac{\dfrac{\vdots\, D'}{\neg A, \Lambda, B(t)}}{\neg A, \Lambda, \exists x\, B(x)}\,\exists R}{\Pi, \Lambda, \exists x\, B(x)}\,cut \quad \text{by} \quad \dfrac{\dfrac{\dfrac{\vdots\, D}{\Pi, A} \qquad \dfrac{\vdots\, D'}{\neg A, \Lambda, B(t)}}{\Pi, \Lambda, B(t)}\,cut}{\Pi, \Lambda, \exists x\, B(x)}\,\exists R$$

Gentzen-style Cut Elimination in the M-Y system

- Permute cut across ε_1 rule:

$$\dfrac{\begin{array}{c}\vdots\, D\\ \Pi, A\end{array} \quad \dfrac{\begin{array}{c}\vdots\, D'\\ \neg A, \Gamma, \Delta(\varepsilon_x B(x)), \neg B(\varepsilon_x B(x))\end{array} \quad \begin{array}{c}\vdots\, D''\\ \Gamma, B(t)\end{array}}{\neg A, \Gamma, \Delta(\varepsilon_x B(x))}\, \varepsilon_1}{\Pi, \Gamma, \Delta(\varepsilon_x B(x))}\, cut$$

replace with

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, D\\ \Pi, A\end{array} \quad \begin{array}{c}\vdots\, D'\\ \neg A, \Gamma, \Delta(\varepsilon_x B(x)), \neg B(\varepsilon_x B(x))\end{array}}{\Pi, \Gamma, \Delta(\varepsilon_x B(x)), \neg B(\varepsilon_x B(x))}\, cut \quad \begin{array}{c}\vdots\, D''\\ \Gamma, B(t)\end{array}}{\Pi, \Gamma, \Delta(\varepsilon_x B(x))}\, \varepsilon_1$$

- Condition on ε_1 is violated if $\neg A$ is Δ.

Gentzen-style Cut Elimination in the M-Y system

- Permute cut across ε_1 rule:

$$\dfrac{\begin{array}{c}\vdots\, D\\ \Pi, A(\varepsilon_x B(x))\end{array} \quad \dfrac{\begin{array}{c}\vdots\, D'\\ \neg A(\varepsilon_x B(x)), \Gamma, \neg B(\varepsilon_x B(x))\end{array} \quad \begin{array}{c}\vdots\, D''\\ \Gamma, B(t)\end{array}}{\neg A(\varepsilon_x B(x)), \Gamma}\,\varepsilon_1}{\Pi, \Gamma}\,cut$$

replace with

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, D\\ \Pi, A(\varepsilon_x B(x))\end{array} \quad \begin{array}{c}\vdots\, D'\\ \neg A(\varepsilon_x B(x)), \Gamma, \neg B(\varepsilon_x B(x))\end{array}}{\Pi, \Gamma, \neg B(\varepsilon_x B(x))}\,cut \quad \begin{array}{c}\vdots\, D''\\ \Gamma, B(t)\end{array}}{\Pi, \Gamma}\,\varepsilon_1$$

- Condition on ε_1 is violated.

Schütte-Tait Style Cut Elimination

- Main induction on cut rank, i.e., complexity of cut formula.
- Induction step: reduce complexity of cut formula.
- For instance, if proof ends in

$$\dfrac{\begin{matrix}\vdots\ D \\ \Pi, \neg(A \wedge B)\end{matrix} \qquad \begin{matrix}\vdots\ D' \\ \Lambda, A \wedge B\end{matrix}}{\Pi, \Lambda}\ cut$$

replace with

$$\dfrac{\dfrac{\begin{matrix}\vdots\ D_1 \\ \Pi, \neg A, \neg B\end{matrix} \qquad \begin{matrix}\vdots\ D'_1 \\ \Lambda, A\end{matrix}}{\Pi, \Lambda, \neg B}\ cut \qquad \begin{matrix}\vdots\ D'_2 \\ \Lambda, B\end{matrix}}{\Pi, \Lambda}\ cut$$

Schütte-Tait Style Cut Elimination: Inversion Lemma

- Requires inversion lemma.
- Typical case: If $D' \vdash \Pi, A \wedge B$ then there is a $D'_1 \vdash \Pi, A$ of cut rank and length $\leq$ that of D'.
- Proof idea: Replace all ancestors of $A \wedge B$ in D' by A and fix rules that get broken.
- For instance, replace

$$\dfrac{\begin{matrix}\vdots \\ \Gamma, A\end{matrix} \quad \begin{matrix}\vdots \\ \Gamma, B\end{matrix}}{\Gamma, A} \wedge R \quad \text{by} \quad \begin{matrix}\vdots \\ \Gamma, A\end{matrix}$$

Schütte-Tait Style Cut Elimination in the M-Y System

- Consider derivation D' which contains ε_1 rule:

$$\dfrac{\begin{matrix}\vdots \\ \Pi, A \wedge B(\varepsilon_x C(x)), \neg C(\varepsilon_x C(x))\end{matrix} \quad \begin{matrix}\vdots \\ \Pi, C(t)\end{matrix}}{\Pi, A \wedge B(\varepsilon_x C(x)} \; \varepsilon_1$$

$(A \wedge B(\varepsilon_x C(x))$ is $\Delta(\varepsilon_x C(x)))$.

- Inversion lemma produces

$$\dfrac{\begin{matrix}\vdots \\ \Pi, A, \neg C(\varepsilon_x C(x))\end{matrix} \quad \begin{matrix}\vdots \\ \Pi, C(t)\end{matrix}}{\Pi, A} \; \varepsilon_1$$

- No longer satisfies condition of ε_1.

Semantics

- Choice functions
- Intensional semantics complete for Hilbert's original system
- Other semantics possible (Blass & Gurevich, Gratzl)
- Linguistic interest, arbitrary objects
- Further work:
  - Generic consequence
  - Semantics for intutionistic systems

Proof Theory

- Epsilon theorem alternative proof theoretic approach
- Herbrand complexity depending ony on critical count
- However:
  - Does not work in intuitionistic logic
  - Does not (yet) combine well with sequent systems
- Further work:
  - Find nice sequent system or prove cut elimination for M-Y
  - Investigate Meyer Viol's natural deduction systems
  - Intuitionistic systems

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