Ramsey Theory on Trees and Applications

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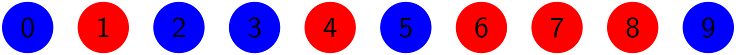
Pigeonhole Principle

If $n > m$ and there are n pigeons and m holes, and each pigeon is put in a hole, then at least one of the holes must contain at least two pigeons.

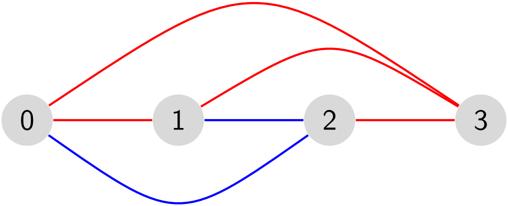
Infinite Pigeonhole Principle

Given a coloring of all the natural numbers $\mathbb{N}$ into red and blue, there is an infinite subset of the natural numbers all of the same color.



Coloring Pairs of Numbers

Finite Ramsey's Theorem. Given $n \geq 2$, there is a number r such that for any set of numbers X of size r and any coloring of the pairs in X into red and blue, there is a subset $Y \subseteq X$ of size n such that each pair of numbers from Y has the same color.



In this example, there is a triple $\{0, 1, 3\}$ such that all pairs in this triple are colored red.

Infinite Ramsey's Theorem

Given a coloring of all the pairs of natural numbers by red and blue, there is an infinite set of natural numbers M such that each pair from M has the same color.

General Ramsey's Theorem

Ramsey's Theorem includes colorings of sets of size larger than 2 and also colorings into more than two colors.

Ramsey's Theorem. Given any $k, l \geq 1$ and a coloring on the collection of all k-element subsets of $\mathbb{N}$ into l colors, there is an infinite set M of natural numbers such that each k-element subset of M has the same color.

Finite Ramsey's Theorem. Given any $k, l, n \geq 1$, there is a number r such that for any set X of size r and any coloring of the k-element subsets of X into l colors, there is a subset $Y \subseteq X$ of size n such that all k-element subsets of Y have the same color.

Ramsey's Theorem has been extended to many types of structures.

Structural Ramsey Theory is the study of looking for a large substructure inside a given structure which is homogeneous for some coloring.

In this talk we will look at Ramsey theorems on trees and their applications to Ramsey theorems on graphs.

Binary Trees

$2 = \{0, 1\}$.

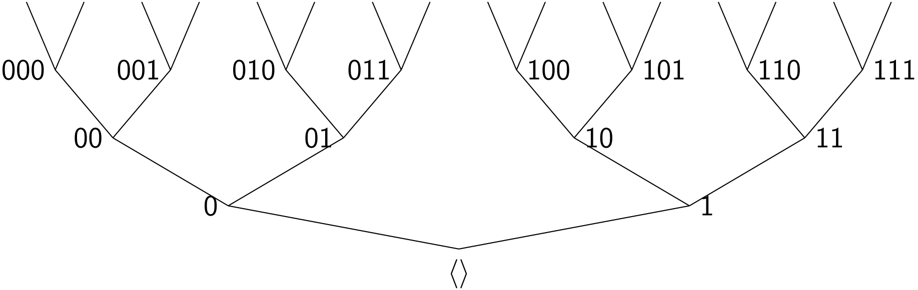
2^n denotes the set of all sequences of 0's and 1's of length n.

$2^{\leq n}$ denotes the set of all sequences of 0's and 1's of length $\leq n$.

$2^{<\omega} = \bigcup_{n<\omega} 2^n$, the set of all finite sequences of 0's and 1's.

A (binary) tree T is a subset of $2^{<\omega}$ such that for any two nodes in T, their meet is also in T.

The Binary Tree of Height Four, $2^{\leq 4}$



Strong Trees

A tree $T \subseteq 2^{<\omega}$ is a strong tree if there is a set of levels $L \subseteq \mathbb{N}$ such that each node in T has length in L, and every node in T branches.

Each strong tree is either isomorphic to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

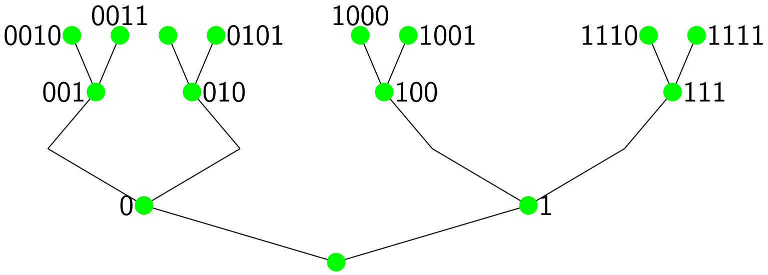


Figure: A strong subtree isomorphic to $2^{\leq 3}$

Halpern-Läuchli Theorem for one tree

Let $T \subseteq 2^{<\omega}$ be an infinite strong tree and suppose the nodes in T are colored red and blue. Then there is an infinite strong subtree $S \subseteq T$ in which all the nodes have the same color.

Halpern-Läuchli Theorem for one tree

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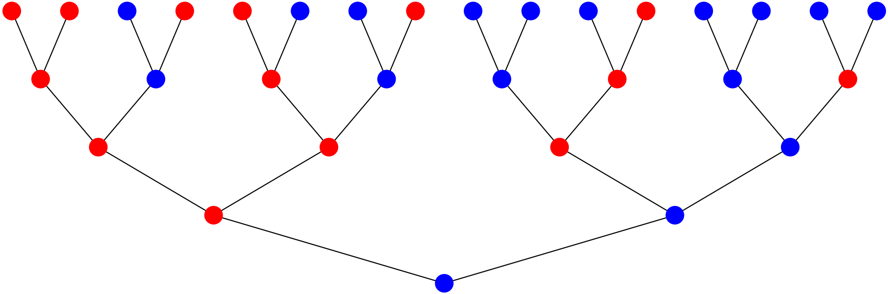
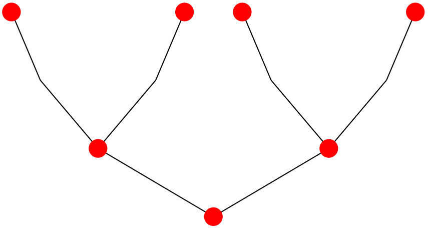


A Monochromatic Strong Subtree Isomorphic to $2^{\leq 2}$



Halpern-Läuchli Theorem for Two Trees

Let T_0, T_1 be infinite strong trees with the same set of levels, L.

For $l \in L$, $T_i(l)$ denotes the members of T_i of length l.

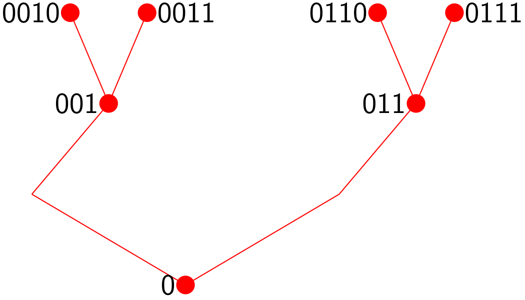
$T_0(l) \times T_1(l)$ denotes the set of all pairs (t_0, t_1) such that $t_0 \in T_0(l)$ and $t_1 \in T_1(l)$.

Halpern-Läuchli Theorem. Let c be a coloring of $\bigcup_{l \in L} T_0(l) \times T_1(l)$ into two colors. Then there are strong subtrees $S_0 \subseteq T_0$ and $S_1 \subseteq T_1$ and an infinite subset $K \subseteq L$ such that S_0 and S_1 are strong trees with levels in K, and c takes only one color on $\bigcup_{k \in K} S_0(k) \times S_1(k)$.

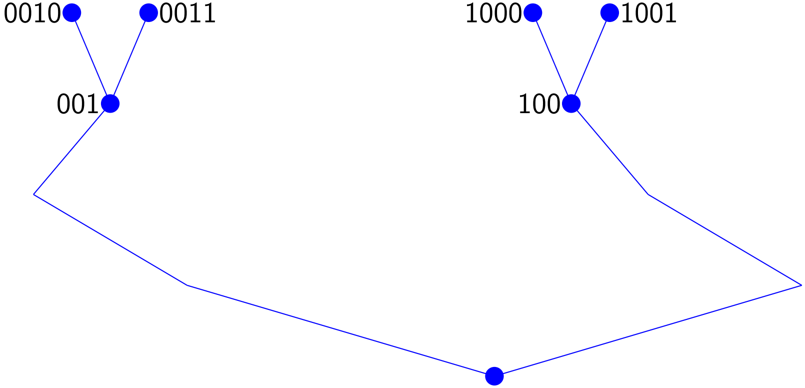
The Halpern-Läuchli Theorem is applied to prove a Ramsey theorem about colorings of all copies of a fixed finite strong tree inside an infinite strong tree.

Recall, all finite strong trees are isomorphic to $2^{\leq k}$, for some $k \geq 0$.

A strong subtree of $2^{\leq 4}$ isomorphic to $2^{\leq 2}$



Another strong subtree of $2^{\leq 4}$ isomorphic to $2^{\leq 2}$



Milliken's Theorem

Let T be an infinite strong tree, $k \geq 0$, and let f be a coloring of all the finite strong subtrees of T which are isomorphic to $2^{\leq k}$.

Then there is an infinite strong subtree $S \subseteq T$ such that all copies of $2^{\leq k}$ in S have the same color.

Remark. For $k = 0$, the coloring is on the nodes of the tree T.

Milliken's Theorem is applied to prove Ramsey theorems for graphs.

Graphs and Ordered Graphs

Graphs are sets of vertices with edges between some of the pairs of vertices.

An ordered graph is a graph whose vertices are linearly ordered.

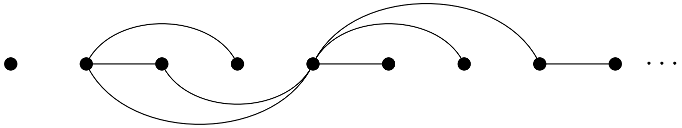


Figure: An ordered graph B

Embeddings of Graphs

An ordered graph A embeds into an ordered graph B if there is a one-to-one mapping of the vertices of A into some of the vertices of B such that each edge in A gets mapped to an edge in B, and each non-edge in A gets mapped to a non-edge in B.

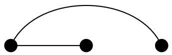


Figure: A

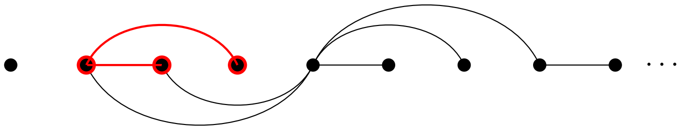
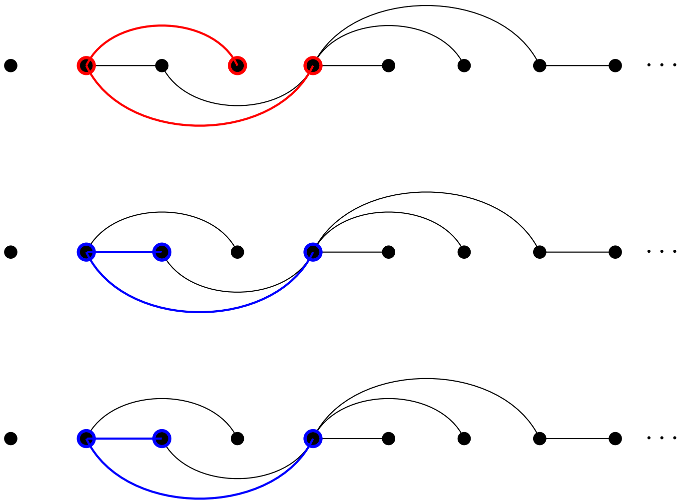
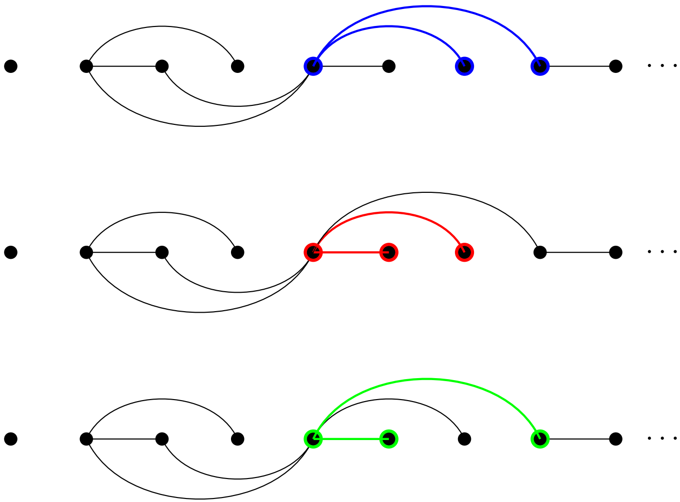


Figure: A copy of A in B

More copies of A into B



Still more copies of A into B



Different Types of Colorings on Graphs

Let G be a given graph.

Vertex Colorings: The vertices in G are colored.

Edge Colorings: The edges in G are colored.

Colorings of Triangles: All triangles in G are colored. (These may be thought of as hyperedges.)

Colorings of *n*-cycles: All *n*-cycles in G are colored.

Colorings of A: Given a finite graph A, all copies of A which occur in G are colored.

Ramsey Theorem for Graphs

Thm. (Nešetřil/Rödl) For any finite ordered graphs A and B such that $\mathrm{A} \le \mathrm{B}$, there is a finite ordered graph C such that for each coloring of all the copies of A in C into red and blue, there is a $\mathrm{B}' \le \mathrm{C}$ which is a copy of B such that all copies of A in B' have the same color.

In symbols, given any $f : \binom{\mathrm{C}}{\mathrm{A}} \to 2$, there is a $\mathrm{B}' \in \binom{\mathrm{C}}{\mathrm{B}}$ such that f takes only one color on all members of $\binom{\mathrm{B}'}{\mathrm{A}}$.

The Random Graph

The random graph is the graph on infinitely many nodes such that for each pair of nodes, there is a 50-50 chance that there is an edge between them.

This is often called the Rado graph since it was constructed by Rado, and is denoted by $\mathcal{R}$.

The random graph is

1. universal for countable graphs: Every countable graph embeds into $\mathcal{R}$.
2. homogeneous: Every isomorphism between two finite subgraphs in $\mathcal{R}$ is extendible to an automorphism of $\mathcal{R}$.

Vertex Colorings in $\mathcal{R}$

Thm. (Folklore) Given any coloring of vertices in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$ which is also a random graph such that the vertices in $\mathcal{R}'$ all have the same color.

Edge Colorings in $\mathcal{R}$

Thm. (Pouzet/Sauer) Given any coloring of the edges in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$ which is also a random graph such that the edges in $\mathcal{R}'$ take no more than two colors.

Can we get down to one color?

No!

The proof that this is best possible uses Ramsey theory on trees and is at the heart of the next theorem.

Colorings of Copies of Any Finite Graph in $\mathcal{R}$

Thm. (Sauer) Given any finite graph A, there is a finite number $n(\mathrm{A})$ such that the following holds:

For any $l \geq 1$ and any coloring of all the copies of A in $\mathcal{R}$ into l colors, there is a subgraph $\mathcal{R}' \leq \mathcal{R}$, also a random graph, such that the set of copies of A in $\mathcal{R}'$ take on no more than $n(\mathrm{A})$ colors.

In the jargon, we say that the big Ramsey degrees for $\mathcal{R}$ are finite, because we can find a copy of the whole infinite graph $\mathcal{R}$ in which all copies of A have at most some bounded number of colors.

The Main Steps in Sauer's Proof

Proof outline:

1. Trees can be used to code graphs.
2. Only diagonal trees need be considered.
3. Each diagonal tree fits uniquely inside a strong tree.
4. There is a Ramsey theorem for strong trees due to Milliken.
5. Extract from Milliken's Theorem one color for each isomorphism type of tree.
6. The number of isomorphism types of trees coding A gives the number $n(\mathrm{A})$.

Using Trees to Code Graphs

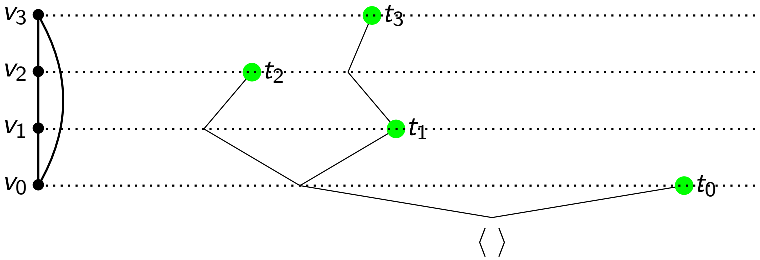
Let A be a graph.

Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

The n-th **coding node** t_n in $2^{<\omega}$ codes v_n.

$$v_n \ E \ v_i \Leftrightarrow t_n(|t_i|) = 1$$

A Tree Coding a 4-Cycle



Diagonal Trees Code Graphs

A tree T is diagonal if there is at most one meet or terminal node per level.

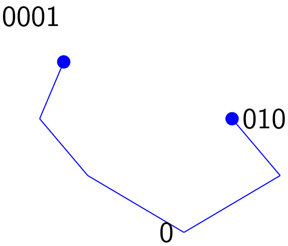


Figure: A diagonal tree D coding an edge between two vertices

Every graph can be coded by the terminal nodes of a diagonal tree.
Moreover, there is a diagonal tree which codes $\mathcal{R}$.

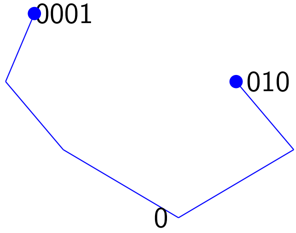
Strong Tree Envelopes of Diagonal Trees

Each diagonal tree is enveloped in a unique minimal strong tree.

This is called the strong tree envelope.

This one-to-one correspondence between a diagonal tree and its strong tree envelope allows us to transfer the coloring of the diagonal tree to an associated coloring of strong trees.

Strong Tree Envelopes of Diagonal Trees



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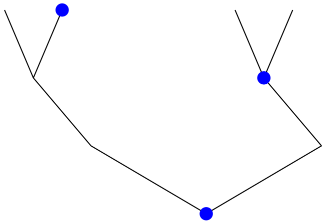
Strong Tree Envelopes of Diagonal Trees



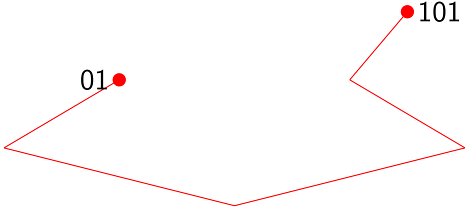
$\langle\ \rangle$

Figure: The strong tree enveloping D

The Big Ramsey Degrees for the Random Graph

The finite Ramsey degree for a given finite graph A is the number of different types of diagonal trees coding A.

There are exactly two types of diagonal trees coding an edge. The tree D a few slides ago, and the following:



Now we turn our attention to Ramsey theory of triangle-free graphs.

Triangle-free graphs

A graph G is triangle-free if no copy of a triangle occurs in G.

In other words, given any three vertices in G, at least two of the vertices have no edge between them.

Nešetřil-Rödl Theorem

Given finite ordered triangle-free graphs $\mathrm{A} \leq \mathrm{B}$, there is a finite ordered triangle-free graph C such that for any coloring of the copies of A in C, there is a copy $\mathrm{B}' \in \binom{\mathrm{C}}{\mathrm{B}}$ such that all copies of A in B' have the same color.

The Universal Triangle-Free Graph

The universal triangle-free graph $\mathcal{H}_3$ is the triangle-free graph on infinitely many vertices into which every countable triangle-free graph embeds.

The universal triangle-free graph is also homogeneous: Any isomorphism between two finite subgraphs of $\mathcal{H}_3$ extends to an automorphism of $\mathcal{H}_3$.

Given any finitely many vertices V in $\mathcal{H}_3$ with no edges between them and any other finite collection of vertices W in $\mathcal{H}_3$, there is another vertex x in $\mathcal{H}_3$ such that x has an edge with every vertex in V and x has no edges with any vertex in W.

The universal triangle-free graph was constructed by Henson in 1971. Henson also constructed universal k-clique-free graphs for each $k \geq 3$.

Vertex Colorings

Theorem. (Komjáth/Rödl) For each coloring of the vertices of $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \leq \mathcal{H}_3$ which is also universal triangle-free in which all vertices have the same color.

Edge Colorings

Theorem. (Sauer) For each coloring of the edges of $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \leq \mathcal{H}_3$ which is also universal triangle-free such that all edges in $\mathcal{H}$ have at most 2 colors.

This is best possible for edges.

Are the big Ramsey degrees for $\mathcal{H}_3$ finite?

What about colorings of finite triangle-free graphs in general?

Are the big Ramsey degrees for $\mathcal{H}_3$ finite?

That is, given any finite triangle-free graph A, is there a number $n(\mathrm{A})$ such that for any l and any coloring of the copies of A in $\mathcal{H}_3$ into l colors, there is a subgraph $\mathcal{H}$ of $\mathcal{H}_3$ which is also universal triangle-free, and in which all copies of A take on no more than $n(\mathrm{A})$ colors?

$\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem*. (D.) For each finite triangle-free graph A, there is a number $n(\mathrm{A})$ such that for any coloring of the copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H}' \le \mathcal{H}_3$ which is also universal triangle-free such that all copies of A in $\mathcal{H}'$ take no more than $n(A)$ colors.

In the jargon, we say that the big Ramsey degrees for the universal triangle-free graph are finite.

$*$ Still to be finished typing up.

Structure of Proof

Develop a notion of strong triangle-free trees coding triangle-free graphs.

These trees have special coding node inside the tree coding the vertices of the graph and branch as much as possible without any branch coding a triangle.

Construct a strong triangle-free tree coding $\mathcal{H}_3$.

Prove a Ramsey theorem for strong triangle-free trees.

(The proofs of these use the set-theoretic method of forcing.)

For each finite triangle-free graph A there are finitely many diagonal trees coding A. Find the correct notion of a triangle-free envelope $E(\mathrm{A})$.

Transfer colorings from diagonal trees to their envelopes.

Apply the Ramsey theorem.

Lastly, take a diagonal subtree which codes $\mathcal{H}_3$ along with a collection of 'witnessing nodes' which are used to construct envelopes. This finishes the proof.

Finite strong triangle-free trees

Finite strong triangle-free trees are trees which code a triangle-free graph and which branch as much as possible, subject to the

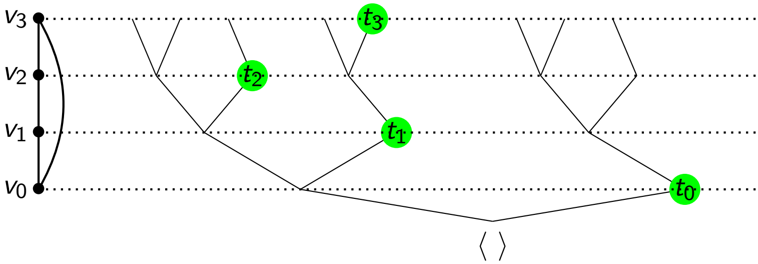
Triangle-Free Extension Criterion: A node t at the level of the n-th coding node t_n extends right if and only if t and t_n have no parallel 1's.

Every node always extends left.

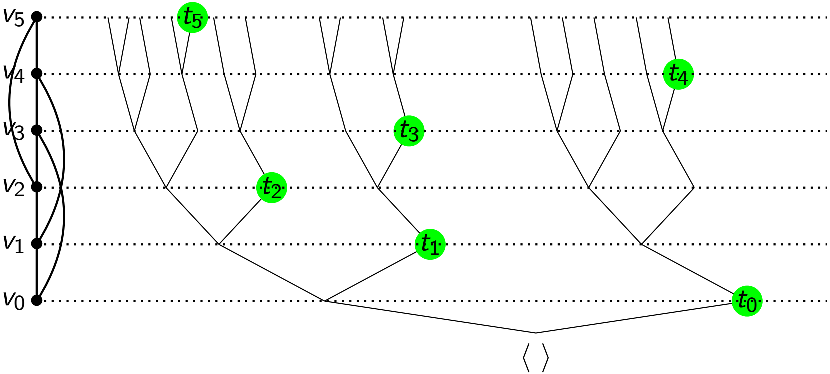
Building a strong triangle-free tree $\mathbb{T}$ to code $\mathcal{H}_3$

Let $\langle F_i : i < \omega \rangle$ be a listing of all finite subsets of $\mathbb{N}$ such that each set repeats infinitely many times.

Alternate taking care of requirement F_i and taking care of density requirement for the coding nodes.



Building a strong triangle-free $\mathbb{T}$ to code $\mathcal{H}_3$



Ramsey theorem for strong triangle-free trees

The tree on the previous slide can be stretched so that it is a diagonal tree still coding $\mathcal{H}_3$ in the same manner.

Theorem. (D.) For each finite diagonal tree D, there is a finite number $n(D)$ such that for any coloring of all copies of D in $\mathbb{T}$, there is a subtree T of $\mathbb{T}$ which is also strong triangle-free and coding $\mathcal{H}_3$ such that the copies of D in T take no more than $n(D)$ colors.

This is an analogue of the Milliken Theorem to the setting of triangle-free graphs.

Concluding Remarks

We've seen Ramsey's Theorem, Halpern-Läuchli Theorem, Milliken Theorem, Ramsey Theorem for Strong Triangle-Free Trees, and applications to solving the problems of big Ramsey degrees for the Rado graph and for the universal triangle-free graph.

There are many other related results we did not go into, for instance, finite sets of rationals (Devlin), related results for uncountable cardinals (Shelah; Džamonja, Larson, Mitchell), questions about bounds for finite structural Ramsey theorems, and many other interplays between Ramsey theorems on trees and other structures.

Directions for More Research

1. Big Ramsey degrees for other infinite structures.
2. Bounds on finite versions of Ramsey theorems for different structures.
3. Recursive analysis or computational complexity of the content of Ramsey theorems for infinite structures.

We hope this has encouraged you to learn more about structural Ramsey Theory and possible applications to other areas of mathematics and science.

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Most graphics in this talk were either made by or modified from codes made by Timothy Trujillo.