Modal logics of Tychonoff HED-spaces

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Common semantics for modal logic

Kripke semantics

A *frame* $\mathfrak{F}$ is a pair (W, R) in which $R \subseteq W \times W$.

Algebraic semantics

A *modal algebra* $\mathfrak{A}$ is a pair $(B, \Diamond)$ where B is a Boolean algebra and $\Diamond : B \to B$ satisfies

$$\begin{aligned} \Diamond(a \vee b) &= \Diamond a \vee \Diamond b, \\ \Diamond 0 &= 0. \end{aligned}$$

$\Diamond$ has a *dual* operation $\Box : B \to B$ given by $\Box a = -\Diamond - a$ satisfying

$$\begin{aligned} \Box(a \wedge b) &= \Box a \wedge \Box b, \\ \Box 1 &= 1. \end{aligned}$$

Interpreting the modal language: Kripke semantics

In $\mathfrak{F} = (W, R)$

Let $\nu(p) \subseteq W$ be the worlds where propositional letter p is true. Truth extends to all formulas as expected for classical connectives and for modal connectives by:

$$w \vDash_\nu \Diamond\varphi \text{ iff } \exists v \in W,\ wRv \text{ and } v \vDash_\nu \varphi,$$

$$w \vDash_\nu \Box\varphi \text{ iff } \forall v \in W \text{ with } wRv,\ v \vDash_\nu \varphi.$$

- φ is *satisfied* in $\mathfrak{F}$ if $\exists \nu,\ \exists w \in W, w \vDash_\nu \varphi$.
- φ is *valid* in $\mathfrak{F}$ if $\forall \nu,\ \forall w \in W,\ w \vDash_\nu \varphi$, denoted $\mathfrak{F} \vDash \varphi$.
- $\mathfrak{F}$ *refutes* φ if $\mathfrak{F} \nvDash \varphi$.
- The *logic of a frame* $\mathfrak{F}$ is $\mathrm{Log}(\mathfrak{F}) = \{\varphi \mid \mathfrak{F} \vDash \varphi\}$.
- The *logic of a class of frames* $\mathcal{C}$ is $\mathrm{Log}(\mathcal{C}) = \bigcap_{\mathfrak{F} \in \mathcal{C}} \mathrm{Log}(\mathfrak{F})$.

Interpreting the modal language: algebraic semantics

In $\mathfrak{A} = (B, \Diamond)$

letters	elements of B
classical connectives	Boolean operations of B
modal diamond	$\Diamond$
modal box	$\Box$

- φ is *satisfied* in $\mathfrak{A}$ if φ evaluates to 1 for some assignment of the letters.
- φ is *valid* in $\mathfrak{A}$ if φ evaluates to 1 for all assignments, denoted $\mathfrak{A} \models \varphi$.
- $\mathfrak{A}$ *refutes* φ if $\mathfrak{A} \not\models \varphi$.
- The *logic of an algebra* $\mathfrak{A}$ is $\mathrm{Log}(\mathfrak{A}) = \{\varphi \mid \mathfrak{A} \models \varphi\}$.
- The *logic of a class of algebras* $\mathcal{C}$ is $\mathrm{Log}(\mathcal{C}) = \bigcap_{\mathfrak{A} \in \mathcal{C}} \mathrm{Log}(\mathfrak{A})$.

Algebraic semantics generalizes Kripke semantics

Let $\mathfrak{F} = (W, R)$ be a frame. For $A \subseteq W$ let

$$\begin{aligned} R^{-1}[A] &= \{w \in W \mid \exists v \in A,\ wRv\}, \\ R[A] &= \{w \in W \mid \exists v \in A,\ vRw\}. \end{aligned}$$

Definition

$\mathfrak{A}_{\mathfrak{F}} := (\wp(W), \Diamond_R)$ is a modal algebra if $\Diamond_R A := R^{-1}[A]$, and hence $\Box_R A = \{w \in W \mid R[w] \subseteq A\}$.

Theorem

$\mathfrak{F} \vDash \varphi$ iff $\mathfrak{A}_{\mathfrak{F}} \vDash \varphi$, for any formula φ.

S4 and closure operators

Recall

- $\mathbf{S4} := \mathbf{K} + p \to \Diamond p + \Diamond\Diamond p \to \Diamond p$.
- Call $\mathfrak{F} = (W, R)$ an **S4**-*frame* if R is reflexive and transitive.
- Call $\mathfrak{A} = (B, \Diamond)$ an **S4**-*algebra* if $\Diamond$ satisfies

$$a \leq \Diamond a \text{ and } \Diamond\Diamond a \leq \Diamond a.$$

  Such $\Diamond$ is called a *closure operator* and its dual $\Box$ an *interior operator*.

Observation

If $\mathfrak{F} = (W, R)$ is an **S4**-frame then $\mathfrak{A}_{\mathfrak{F}} = (\wp(W), \Diamond_R)$ is an **S4**-algebra; that is, $\Diamond_R$ is a closure operator.
There are other closure operators on $\wp(W)$ that are not realized via $\Diamond_R$!

Topological semantics

Definition

Given $\mathfrak{A}_X = (\wp(X), \mathbf{C})$ an **S4**-algebra for a set X ...

- If $\mathbf{C}A = A$, call A *closed*;
- If $\mathbf{I}A = A$, call A *open* (here $\mathbf{I}$ is dual to $\mathbf{C}$);
- $\tau := \{\mathbf{I}A \mid A \subseteq X\}$ is a *topology* for X;
- (X, τ) is a *topological space*. Drop τ if clear.

Topological semantics

- Interpret φ in (X, τ) by interpreting it in $\mathfrak{A}_X$.
- $\Diamond$ is closure, $\Box$ is interior, and formulas are subsets of X.
- φ is *valid* in X if $\mathfrak{A}_X \vDash \varphi$; denoted $X \vDash \varphi$.
- The *logic of a space* X is $\mathrm{Log}(X) = \{\varphi \mid X \vDash \varphi\}$.
- The *logic of a class of spaces* $\mathcal{C}$ is $\mathrm{Log}(\mathcal{C}) = \bigcap_{X \in \mathcal{C}} \mathrm{Log}(X)$.

Semantical hierarchy

Observations

Kripke semantics < Topological semantics < Algebraic semantics

- **S4**-frames correspond to special topological spaces called *Alexandroff* spaces.
- If $\mathfrak{F} = (W, R)$ is an **S4**-frame, then $\tau_R := \{R[A] \mid A \subseteq W\}$ is a topology making (W, τ_R) an Alexandroff space whose closure is R^{-1}.
- If (X, τ) is a space then setting xRy iff $x \in \mathbf{C}\{y\}$ gives (X, R) is an **S4**-frame. When (X, τ) is Alexandroff $\tau = \tau_R$.

Historical development

Topological then algebraic then frame.

Topological semantics was introduced by McKinsey and Tarski.

Origins of the program and progress: I

McKinsey-Tarski theorem (1944)

S4 is the logic of any separable dense-in-itself metric space.

- **S4** is the logic of the class of all topological spaces.
- **S4** is the logic of ‘nice’ spaces.

Drop a hypothesis (Rasiowa & Sikorski 1963)

S4 is the logic of any dense-in-itself metric space.

Kremer’s theorem (2013)

S4 is strongly complete for any dense-in-itself metric space.

Origins of the program and progress: II

Drop another hypothesis (BG&LB 2015)

The modal logics arising from metric spaces form the chain

$$\mathbf{S4} \subset \mathbf{S4.1} \subset \mathbf{S4.Grz} \subset \cdots \subset \mathbf{S4.Grz}_n \subset \cdots \subset \mathbf{S4.Grz}_1$$

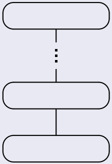
Generalizing from metric spaces

- Metric spaces are subspaces of compact Hausdorff spaces.
- This also true of certain non-metric spaces.
  Tychonoff characterized such spaces which now bear his name; aka completely regular spaces or $T_{3\frac{1}{2}}$-spaces.
- Challenging open problem: Axiomatize the logics arising from Tychonoff spaces.
- We solve this problem for logics extending **S4.3**!

S4.3 and its frames

Recall

- **S4.3** $:=$ **S4** $+ \Box(\Box p \to q) \vee \Box(\Box q \to p)$.
- An **S4**-frame $\mathfrak{F}$ is *rooted* if $\exists r \in W,\ R[r] = W$.
- Call a rooted **S4.3**-frame $\mathfrak{F} = (W, R)$ a *quasi-chain*: satisfies wRv or vRw for any $w, v \in W$; a quasi-chain is a chain of clusters.



Bull-Fine theorem (1966/1971)

There are countably many extensions of **S4.3** and each is finitely axiomatizable and has the finite model property (fmp).

Key ingredient

p-morphic image

$\mathfrak{G} = (V, S)$ a *p-morphic image* of $\mathfrak{F} = (W, R)$ if there is an onto $f : W \to V$ satisfying

- wRu implies $f(w)Sf(u)$, and
- $f(w)Sv$ implies $\exists u \in R[w],\ f(u) = v$.

Useful results about p-morphic images

Let $f : \mathfrak{F} \to \mathfrak{G}$ be an onto p-morphism, then ...

- $\mathfrak{F} \vDash \varphi$ implies $\mathfrak{G} \vDash \varphi$. (preserve validity)
- $\mathfrak{G} \nvDash \varphi$ implies $\mathfrak{F} \nvDash \varphi$. (reflect refutation)
- $f^{-1}(S^{-1}[A]) = R^{-1}[f^{-1}(A)]$ for all $A \subseteq V$.

Axiomatizing extensions of **S4.3**

Partially order the set of all finite quasi-chains $\mathcal{Q}$ by setting

$\mathfrak{F} \leq \mathfrak{G}$ iff $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$.†

Theorem

The lattice of extensions of **S4.3** and lattice of downsets of $(\mathcal{Q}, \leq)$ are dually isomorphic (there is an order reversing bijection).

Proof sketch

For $L \supseteq$ **S4.3** define the mapping $L \mapsto \mathcal{F}_L := \{\mathfrak{F} \in \mathcal{Q} \mid \mathfrak{F} \vDash L\}$.
Surjective: For a downset D, take $L =$ **S4.3** $+ \neg\chi_{\mathfrak{F}}$ where $\chi_{\mathfrak{F}}$ is the Jankov-Fine formula for each minimal $\mathfrak{F} \in \mathcal{Q} \setminus D$. It follows by Fine's theorem (1974) that $D = \mathcal{F}_L$ since for finite quasi-chains we have: $\mathfrak{G} \vDash \neg\chi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is not a p-morphic image of $\mathfrak{G}$.†

So, $L \supseteq$ **S4.3** can be axiomatized as **S4.3** $+ \neg\chi_{\mathfrak{F}}$ for each minimal $\mathfrak{F} \in \mathcal{Q} \setminus \mathcal{F}_L$ (of which there are only finitely many)!

Motivation

Definition

The Zeman formula is zem $= \Box\Diamond\Box p \to (p \to \Box p)$.
The Zeman logic is $\mathbf{S4.Z} = \mathbf{S4} + \mathrm{zem}$.

Theorem

Let $\mathfrak{F} = (W, R)$ be an **S4**-frame. Then
$\mathfrak{F} \vDash \mathrm{zem}$ iff there are no $w, v, u \in W$ with $wRvRu$ satisfying both $w \neq v$ and $\neg(uRv)$.

Theorem

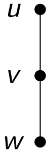
S4.Z has the fmp.

Observation

A finite rooted **S4.Z**-frame $\mathfrak{F}$ must be either a cluster or have a unique root!

Permitted and restricted arrangements in a frame validating zem

$\mathfrak{F} \vDash$ zem iff there are no $w, v, u \in W$ with $wRvRu$ satisfying both $w \neq v$ and $\neg(uRv)$.



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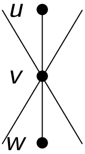

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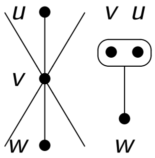

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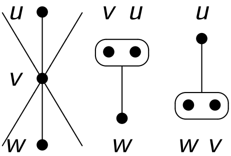

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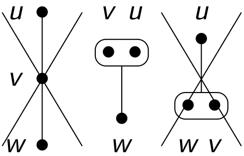

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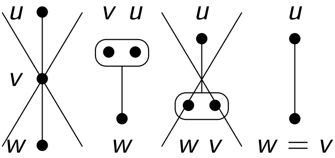
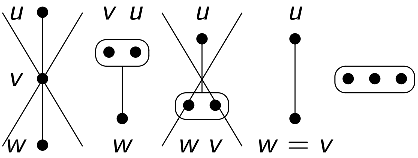


Permitted and restricted arrangements in a frame validating zem

$\mathfrak{F} \models$ zem iff there are no $w, v, u \in W$ with $wRvRu$ satisfying both $w \neq v$ and $\neg(uRv)$.



Permitted and restricted arrangements in a frame validating zem

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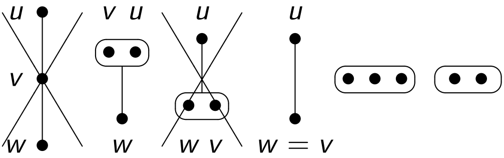
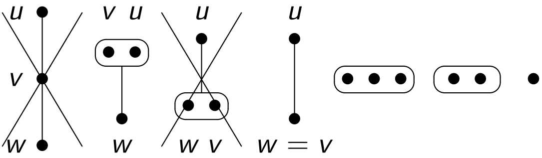


Permitted and restricted arrangements in a frame validating zem

$\mathfrak{F} \models$ zem iff there are no $w, v, u \in W$ with $wRvRu$ satisfying both $w \neq v$ and $\neg(uRv)$.



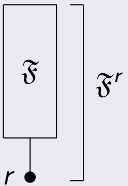
Adding a new root

Recall $\mathcal{Q}$ is set of finite quasi-chains.

Definition

Call $\mathfrak{F} \in \mathcal{Q}$ ...

- *uniquely rooted* if $\mathfrak{F}$ has exactly one root.
- *non-uniquely rooted* if $\mathfrak{F}$ has more than one root.
- Let $\mathfrak{F}^r \in \mathcal{Q}$ denote the uniquely rooted quasi-chain formed by 'adding' a new root 'underneath' $\mathfrak{F}$.



Zemanian logic

Definition

Let $L \supseteq \mathbf{S4.3}$ be consistent. Recall $\mathcal{F}_L = \{\mathfrak{F} \in \mathcal{Q} \mid \mathfrak{F} \vDash L\}$.
Call L *Zemanian* if $\mathfrak{F}^r \in \mathcal{F}_L$ for every non-uniquely rooted $\mathfrak{F} \in \mathcal{F}_L$.

Examples of Zemanian logics

- **S4.3**.
- **S4.3.Z** := **S4.3** + zem.
- **Grz.3** := **S4.3** + $\Box(\Box(p \to \Box p) \to p) \to p$.

Examples of non-Zemanian logics

- **S5**.
- $\mathbf{S4.3}_n$:= **S4.3** + bd_n where:

  $\mathrm{bd}_1 := \Diamond\Box p_1 \to p_1$, and
  $\mathrm{bd}_{n+1} := \Diamond(\Box p_{n+1} \wedge \neg \mathrm{bd}_n) \to p_{n+1}$ for $n \geq 1$.

Properties of Zemanian logics

Definition

Let $(P, \leq)$ be a partial order. Call $S \subseteq P$ *cofinal* if $\forall p \in P,\ \exists s \in S,\ p \leq s$.

Lemma

Let $\mathcal{U}_L = \{\mathfrak{F} \in \mathcal{F}_L \mid \mathfrak{F} \text{ is uniquely rooted}\}$ for consistent $L \supseteq$ **S4.3**. Then L is Zemanian iff $\mathcal{U}_L$ is cofinal in $\mathfrak{F}_L$.

Lemma

A Zemanian logic is the logic of the class of its uniquely rooted quasi-chains.

Modal definability: example I

Let X be a space.

Definition

Call $A \subseteq X$ *nowhere dense* if $\mathbf{IC}A = \varnothing$.
Call X *nodec* if every nowhere dense set is closed; equivalently closed and discrete.

Theorem

Let X be a space and $\mathfrak{F} = (W, R)$ be an **S4**-frame.

- X is nodec iff $X \vDash$ zem.
- $\mathfrak{F} \vDash$ zem iff there are no $w, v, u \in W$ with $wRvRu$ with $w \neq v$ and $\neg(uRv)$.
- **S4.Z** is the logic of the class of all nodec spaces.

Modal definability: example II

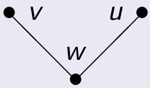
Definition

A space X is *extremally disconnected* (ED) if $\forall U \in \tau,\ \mathbf{C}U \in \tau$.

Theorem

Let X be a space and $\mathfrak{F} = (W, R)$ be an **S4**-frame.

- X is ED iff $X \models \Diamond\Box p \to \Box\Diamond p$.
- $\mathfrak{F} \models \Diamond\Box p \to \Box\Diamond p$ iff $\forall w, v, u \in W$, if wRv and wRu then



Modal definability: example II

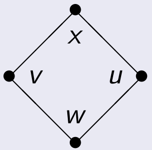
Definition

A space X is *extremally disconnected* (ED) if $\forall U \in \tau,\ \mathbf{C}U \in \tau$.

Theorem

Let X be a space and $\mathfrak{F} = (W, R)$ be an **S4**-frame.

- X is ED iff $X \models \Diamond\Box p \to \Box\Diamond p$.
- $\mathfrak{F} \models \Diamond\Box p \to \Box\Diamond p$ iff $\forall w, v, u \in W$, if wRv and wRu then $\exists x \in W$, vRx and uRx.



- **S4.2** $:=$ **S4** $+ \Diamond\Box p \to \Box\Diamond p$ is the logic of the class of all ED-spaces.

Modal definability: example III

Definitions

Let (X, τ) be a space and $Y \subseteq X$.

- Set $\sigma = \{U \cap Y \mid U \in \tau\}$, then (Y, σ) is a *subspace* of X.
- Call X *Hereditarily extremally disconnected* (HED) if every subspace of X is ED.

Theorem

Let X be a space and $\mathfrak{F} = (W, R)$ be an **S4**-frame.

- X is HED iff $X \vDash \Box(\Box p \to q) \vee \Box(\Box q \to p)$.
- $\mathfrak{F} \vDash \Box(\Box p \to q) \vee \Box(\Box q \to p)$ iff every subframe of $\mathfrak{F}$ validates $\Diamond\Box p \to \Box\Diamond p$.
- **S4**.3 is the logic of the class of all HED-spaces.

Important tools

Let (X, τ) and (Y, σ) be topological spaces.

Validity preserving operations

- Open subspace: $Y \in \tau$ and $\sigma = \{U \cap Y \mid U \in \tau\}$.
- Topological sum $X \oplus Y$: $X \cup Y$ with $\{U \cup V \mid U \in \tau, V \in \sigma\}$ as open subsets (assuming X and Y are disjoint).
- Interior image: there is onto $f : X \to Y$ that is interior; i.e.
  - f is continuous: $f^{-1}(U) \in \tau$ if $U \in \sigma$.
  - f is open: $f(U) \in \sigma$ if $U \in \tau$.

f is interior iff $f^{-1}(\mathbf{C}_Y A) = \mathbf{C}_X f^{-1}(A)$ for all $A \subseteq Y$
iff $f^{-1}(\mathbf{I}_Y A) = \mathbf{I}_X f^{-1}(A)$ for all $A \subseteq Y$.

Basic results

Let $X \neq \varnothing$ be a space and $\mathfrak{F} = (W, R)$ be a finite rooted **S4**-frame.

Lemma: topological version of Fine's theorem

$X \vDash \neg\chi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is not an interior image of any open subspace of X.

Corollary

If $\mathfrak{F} \vDash \mathrm{Log}(X)$ then $\mathfrak{F}$ is an interior image of some open subspace of X.

Lemma

If X is T_1 (hence Tychonoff) and $\mathfrak{F} \in \mathcal{Q}$ is non-uniquely rooted, then $\mathfrak{F}$ is an interior image of X iff $\mathfrak{F}^r$ is an interior image of X.

Technique

Main theorem

Let $L \supseteq$ **S4.3** be consistent.
L is Zemanian iff L is the logic of a Tychonoff HED-space X.

Idea of proof: $\Leftarrow$

Apply previous slide.

Idea of proof: $\Rightarrow$

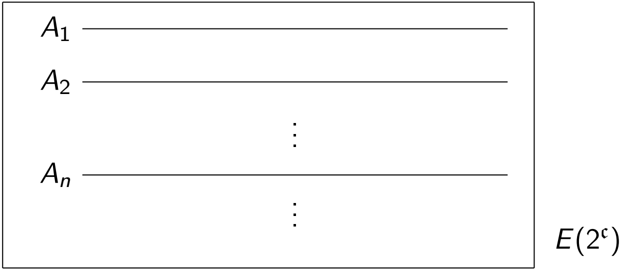
- Build a Tychonoff HED-space $X_{\mathfrak{F}}$ for each uniquely rooted $\mathfrak{F} \in \mathcal{Q}$ of depth > 1 such that $\mathrm{Log}(X_{\mathfrak{F}}) = \mathrm{Log}(\mathfrak{F})$; requires ...
  - $\mathfrak{F}$ is an interior image of $X_{\mathfrak{F}}$: depth and cluster size!
  - If $\mathfrak{G} \in \mathcal{Q}$ is an interior image of an open subspace of X, then $\mathfrak{G}$ is a p-morphic image of $\mathfrak{F}$; that is $\mathfrak{G} \leq \mathfrak{F}$.
- Take X to be the topological sum of $X_{\mathfrak{F}}$ where $\mathfrak{F} \in \mathcal{U}_L$ of depth > 1.

Building $X_{\mathfrak{F}}$: basic ingredients

$E(2^{\mathfrak{c}})$

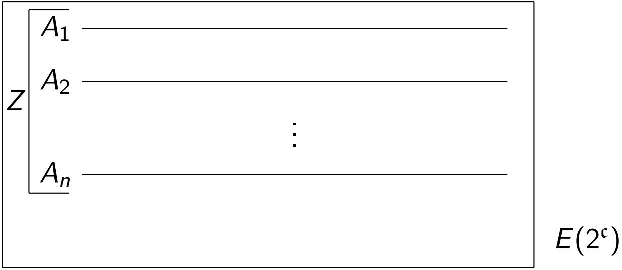
$2 :=$ two point discrete space (every set is open)
$2^{\mathfrak{c}} :=$ Cantor cube; product space of continuum many copies of 2
$E(2^{\mathfrak{c}}) :=$ Gleason cover of the Cantor cube $2^{\mathfrak{c}}$; Stone space of Boolean algebra of regular open subsets of $2^{\mathfrak{c}}$

Building $X_{\mathfrak{F}}$: basic ingredients



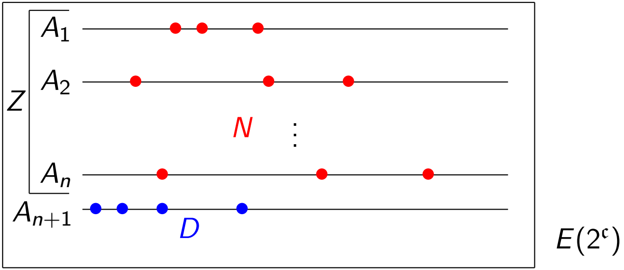
There are pairwise disjoint $A_1, A_2, \ldots, A_n, \cdots \subseteq E(2^{\mathfrak{c}})$ such that each A_i is ...
countable, dense, ED, hereditarily irresolvable, (dense-in-itself) and nodec;
A_i and A_j are far when $i \neq j$.

BUILDING $X_{\mathfrak{F}}$: BASIC INGREDIENTS



Let $Z = \bigcup_{i=1}^{n} A_i$ be a subspace of $E(2^{\mathfrak{c}})$. Then ...
Z is nodec and
every open subset $U \subseteq Z$ is n-resolvable and $(n+1)$-irresolvable.

Building $X_{\mathfrak{F}}$: basic ingredients

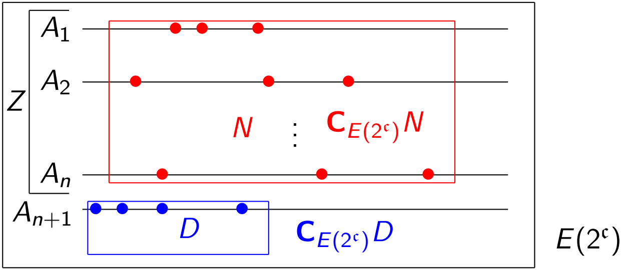


Let $Z = \bigcup_{i=1}^{n} A_i$ be a subspace of $E(2^{\mathfrak{c}})$. Then …

Z is nodec and for any nowhere dense set $N \subseteq Z$ and any (infinite) discrete set $D \subseteq A_{n+1}$

every open subset $U \subseteq Z$ is n-resolvable and $(n+1)$-irresolvable.

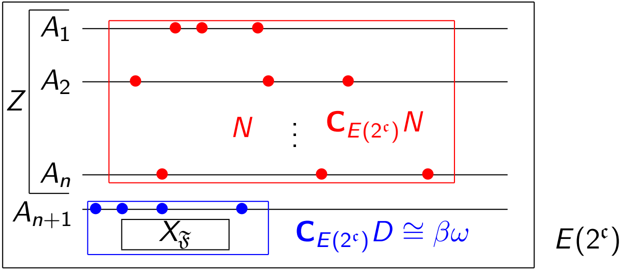
Building $X_{\mathfrak{F}}$: basic ingredients



Let $Z = \bigcup_{i=1}^{n} A_i$ be a subspace of $E(2^{\mathfrak{c}})$. Then ...

Z is nodec and for any nowhere dense set $N \subseteq Z$ and any (infinite) discrete set $D \subseteq A_{n+1}$, $\mathbf{C}_{E(2^{\mathfrak{c}})}(N) \cap \mathbf{C}_{E(2^{\mathfrak{c}})}(D) = \varnothing$ and ...
every open subset $U \subseteq Z$ is n-resolvable and $(n+1)$-irresolvable.

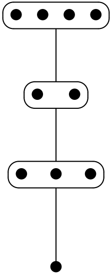
Building $X_{\mathfrak{F}}$: basic ingredients



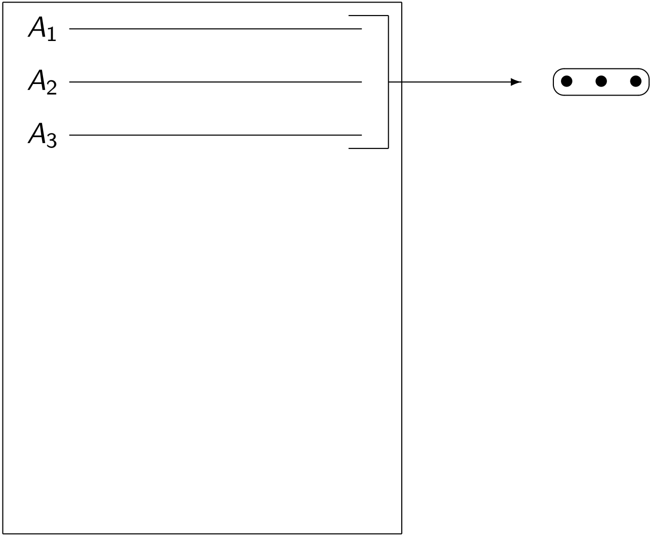
Since $\mathbf{C}_{E(2^{\mathfrak{c}})}(D) \cong \beta\omega$, it follows from Efimov's theorem that any $X_{\mathfrak{F}}$ we build embeds in $\mathbf{C}_{E(2^{\mathfrak{c}})}(D)$.

An example

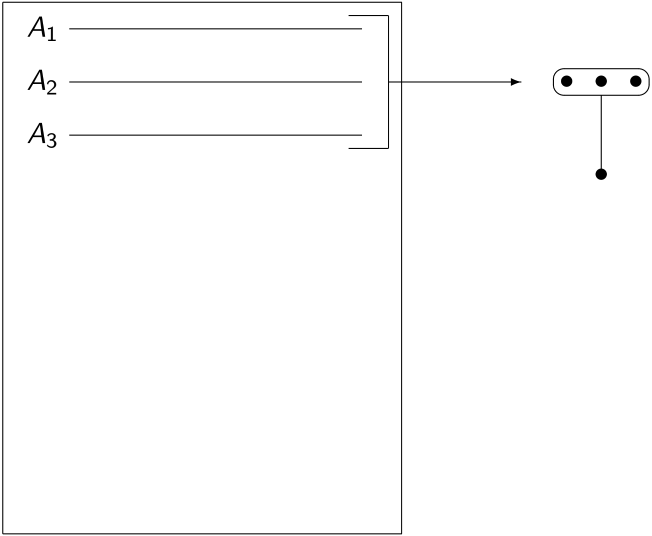
We build $X_{\mathfrak{F}}$ where $\mathfrak{F}$ is ...



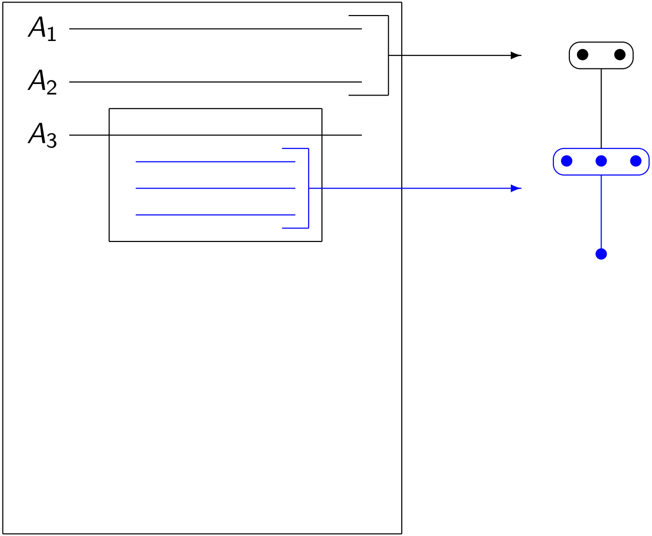
An example



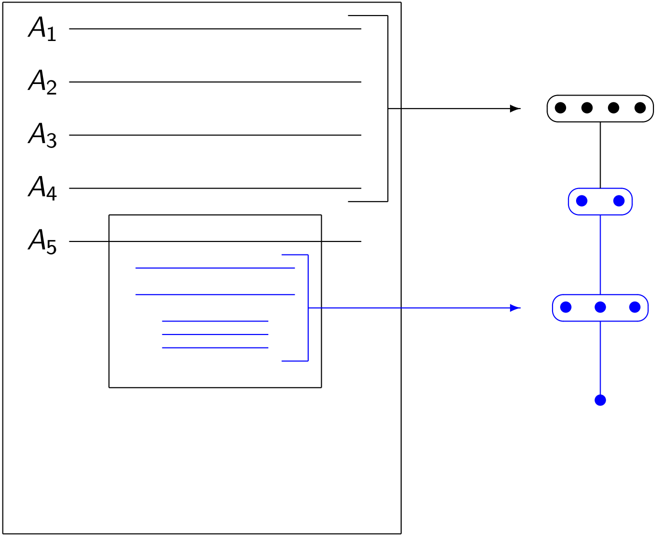
An example



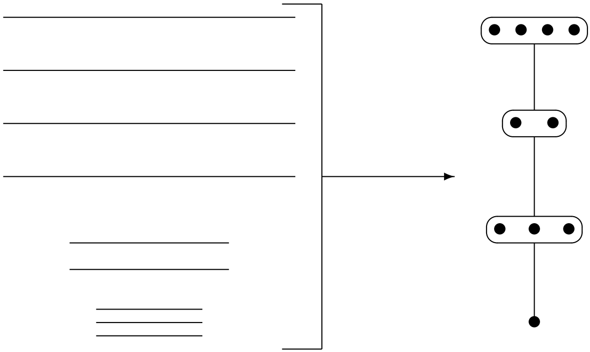
An example



An example



An example



Thank you ...

Both organizers and audience.

Any questions?